RESONANCES FOR AXIOM A FLOWS

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Abstract

Given an Axiom A flow on $M$ and smooth functions $B, C: M \to \mathbb{R}$, we show that the time correlation function $\rho_{BC}$ for a Gibbs state $\rho$ has a Fourier transform $\hat{\rho}_{BC}$ meromorphic in a strip. This complements a result by Pollicott [7]. The residues of the poles of $\hat{\rho}_{BC}$ are investigated. In the simplest case, they have the form $\sigma^{-1} (B)^{-1} (C)$ where $\sigma^{-1}, \sigma^1$ are Gibbs distributions, i.e., (Schwartz) distributions on $M$ further specified in the paper. This is a companion to an earlier paper [9] where similar results have been obtained for Axiom A diffeomorphisms.

0. Introduction

In an earlier paper [9] we have studied the time correlation functions for Axiom A diffeomorphisms. These correlation functions have Fourier transforms which are meromorphic in a strip, and we have identified the residues of the poles in that strip in terms of Gibbs distributions. In the present paper we obtain a similar result for Axiom A flows.

Let $(f^t)$ be a $C^{1+\varepsilon}$ Axiom A flow on a compact manifold $M$ (which we may take as $C^\infty$). We assume that $\rho$ is a Gibbs measure on a nontrivial basic set $\Lambda$ (see Bowen and Ruelle [4]) and let $B, C$ be smooth real functions on $M$. Define the correlation function

$$\rho_{BC}(t) = \int \rho(dx) \mathbb{B}(f^t x) C(x) - \left[ \int \rho(dx) \mathbb{B}(x) \right] \left[ \int \rho(dx) C(x) \right]$$

and its Fourier transform

$$\hat{\rho}_{BC}(\omega) = \int_0^\infty e^{i\omega t} \rho_{BC}(t) \, dt$$

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$^1$The basic set $\Lambda$ is nontrivial if it is not a fixed point or a periodic orbit.
(called the power spectrum if $B = C_0$). Completing an argument of Pollicott [7] we shall show that the function $\rho_{R\infty}$ is meromorphic in a strip $|\text{Im} z| < \theta^*$ (see Theorem 4.1). The poles of $\rho_{R\infty}$ are called resonances, and we shall study their residues. For simplicity we shall consider only simple poles and make a further nondegeneracy assumption which is generically satisfied. Under these conditions, the residues are of the form $\sigma^* (B) = \sigma^* (C)$, where $\sigma^*$ and $\sigma^*$ are Gibbs distributions (see Theorem 4.2). The Gibbs distributions are distributions in the sense of Schwartz on $M$, which will be further specified below.

We refer the reader to Smale [10] and Bowen [1] for a general discussion of Axiom A flows and their basic sets. For the purposes of the present paper we shall essentially use the existence of symbolic dynamics as proved by Bowen [2]. Roughly speaking, symbolic dynamics is obtained by placing in the manifold $M$ a certain number of pieces of hypersurfaces $\Sigma$ transversal to the flow; a point $x$ of a basic set $\Lambda \subset M$ is then specified by the sequence of intersections of its orbit $(f^n x)$ with the $\Sigma$'s.

In the next sections we describe the formal structure of symbolic dynamics (insofar as is needed). This structure is given by the construction of a space $\Omega^\infty$, with a flow $(\tau_\theta)$, and a map $\omega: \Omega^\infty \to M$ such that $\omega \Omega^\infty$ is a basic set $\Lambda$ for the flow $(f^n)$ and $\omega \tau_\theta = f^\theta$.

1. Symbolic dynamics: the shift $\tau$

Let $\mathcal{J}$ be a nonempty finite set, and $(\xi_\ell)$ a square matrix indexed by $\mathcal{J} \times \mathcal{J}$, with elements 0 or 1. (The elements $j$ of $\mathcal{J}$ correspond to the indices of the pieces of hypersurfaces $\Sigma$ mentioned in the introduction; $\xi_j = 1$ if an orbit $(f^n x)$ may successively cross $\Sigma_j$ and $\Sigma_k$.) We define $\Omega$ to be the space of sequences $(\xi_\ell)_{\ell \in \mathcal{J}}$ of elements of $\mathcal{J}$ such that $\xi_{\ell_\ell} = 1$ for all $\ell$. The space $\Omega$ is compact with respect to the topology of pointwise convergence. The shift $\tau: \Omega \to \Omega$ is defined by $(\tau_\theta \xi_\ell) = \xi_{\ell+1}$; $\tau$ is a homeomorphism. The pair $(\Omega, \tau)$ is called a subshift of finite type. We assume that all matrix elements of $\tau$ are $> 0$ for sufficiently large $N$. (This means that $\tau$ is topologically mixing on $\Omega$, which can always be achieved in the present situation.)

Given $X \subset \mathcal{J}$ and $\xi \in \Omega$, we let $\pi_X \xi = (\xi_\ell)_{\ell \in \mathcal{J}}$ be the sequence obtained by restriction of the index set $\mathcal{J}$ to $X$. We also write $\pi_X \Omega = \Omega_X$.

If $A \in \mathcal{W}(\Omega, \mathcal{L})$ we define

$$
\|A\|_\infty = \max \{ |A(\xi)| : \xi \in \Omega \},
$$

$$
\text{var}_\ell A = \sup \{|A(\xi) - A(\xi')| : \xi = \xi' \text{ for } |\ell| < n \},
$$

$$
\|A\|_2 = \sup \theta^{-\text{var}_\ell A}, \quad \text{where } 0 < \theta < 1,
$$

$$
\|A\| = \|A\|_\infty + \|A\|_2.
$$
We let $\psi_A$ be the Banach space of those $A$ for which $\lim_{n \to \infty} \theta^n \text{ var}_A A = 0$, with the norm $\|\cdot\|$. Note that $\psi_A$ is a Banach algebra (i.e. $\|AB\| \leq \|A\| \|B\|$). If $X \subset \mathcal{L}$, we let

$$\psi(X) = \{ A \in \mathcal{W}(\Omega, \mathbb{C}) : A = \pi_1 \in \psi_\pi \}.$$  

This is a Banach space with respect to the induced norm $A \mapsto \|A + \pi_1\|$. We denote by $\mathcal{M}_\pi, \mathcal{M}(X)^* \psi_\pi$ the duals of $\psi_\pi, \psi(X)$ for $\pi$. For $\pi \in \mathcal{M}_\pi$ or $\mathcal{M}(X)^*$ it will be convenient to write

$$\sigma(A) = \int \sigma(\psi_A) A(x)$$

as if $\pi$ were a measure.

The pressure of $A \in \mathcal{W}(\Omega, \mathbb{R})$ is

$$P(A) = \max \{ h(\pi) + \sigma(A) \} \pi \text{ is a } \tau \text{-invariant probability measure},$$

where $h(\pi)$ is the energy of $\pi$ (Kolmogorov-Sinai invariant). If $A \in \psi_{\pi}$, the maximum is reached for a unique measure $\rho$ called the Gibbs state for $A$.

The theory of Gibbs states is discussed in Bowen [3] and Ruelle [8]. In [9] an extension of Gibbs distributions is given (these are elements of $\mathcal{M}_\pi$, not necessarily measures). We shall quote results from the above references as needed. Here we reproduce some definitions of [9] with slightly different notation.\footnote{In particular, it is convenient to write $A_\pi$ instead of $A$ for purposes of later reference.}

If $A_{\pi} \in \mathcal{M}_\pi$, we may introduce an interaction $\Phi$ such that

$$A_{\pi}(\xi) = A_{\pi}(\xi) = -\Phi_{\pi}(\xi_0) + \sum_{k=1}^{\infty} \Phi_{\pi}(\xi_k, \cdots, \xi_k),$$

where $|\Phi_{\pi}| < \text{ const } \theta^n$ (we write $\Phi_{\pi} = 0$ if $k$ is odd). We then define $A_{\pi} \in \psi_{\pi}(-\infty, 0)$ by

$$A_{\pi}(\xi) = -\sum_{k=0}^{-\infty} \Phi_{\pi}(\xi_k, \cdots, \xi_k).$$

Finally we let $A_{\pi}$ be the operator on $\psi_{\pi}(-\infty, 0)$ such that

$$A_{\pi}(\xi) = \sum_{n \leq 0} \mathcal{L}(\xi) \exp(A_{\pi}(\xi^n \circ \eta)) \phi(\eta^\prime \circ \eta),$$

where $\xi \circ \eta = (\cdots, \eta_1, \xi_0, \cdots)$ when $\xi_0 = 1$ (otherwise $\xi \circ \eta$ is undefined). The adjoint $A_{\pi}^*$ acts on $\psi_{\pi}(-\infty, 0)^*$. The spectrum of $A_{\pi}^*$ and $A_{\pi}$ is contained in the disk $\{ z : |z| < \exp(P(\mathcal{E} A_\pi)) \}$, and the part in $\{ z : |z| > \exp(P(\mathcal{E} A_\pi)) \}$ is discrete, consisting of eigenvalues of finite multiplicity.
If \( A_\rho \) is real, \( \exp P(A_\rho) \) is a simple eigenvalue of \( \mathcal{L}_\rho^* \) and \( \mathcal{L}_\rho^* \), and there is no other eigenvalue with the same modulus. Let \( S \) and \( S^* \) be the eigenvectors of \( \mathcal{L}_\rho^* \) and \( \mathcal{L}_\rho^* \) corresponding to \( \exp P(A_\rho) \). Then \( S \) and \( S^* \) is (up to normalization) the image by \( \pi_{-\infty, \theta} \) of the Gibbs state \( \rho \).

For \( A_\rho \) not necessarily real, let \( \lambda, \mu \) be any eigenvalues of \( \mathcal{L}_\rho^* \) and \( \mathcal{L}_\rho^* \) with modulus \( > \theta \exp P(Re A_\rho) \), and let \( S_\lambda, S_\mu \) be the corresponding generalized eigenspaces of \( \mathcal{L}_\rho \), \( \mathcal{L}_\rho^* \). Then the Gibbs distributions on \( \mathcal{H} \) have images by \( \pi_{-\infty, \theta} \) of the form \( S_{\lambda} S^*_{\mu} \) or linear combinations of such products (a precise description is given in \([9]\)).

Let us write

\[
(1.4) \quad d(ze^{i\epsilon}) = \exp \left[ \sum_{n=1}^{\infty} \frac{\xi_n}{n^2} \sum_{\xi \neq 0} \exp(A_n(\xi) + A_n(\xi^t) + \cdots + A_n(\xi^{n-1})) \right].
\]

Then this series converges when \( |z|^2 \theta \exp P(Re A_\rho) < 1 \), with \( \theta < 1 \) as in \([9]\). In this region, the zeros of \( z \rightarrow d(ze^{i\epsilon}) \) coincide with the inverses \( \lambda^{-1} \) of the eigenvalues of \( \mathcal{L}_\rho \), and have the same multiplicity.

2. Symbolic dynamics: the flow \( \mathcal{H}_s \)

Consider the compact set \( \Omega \times [0,1] \) and identify \((\xi, 0)\) with \((\tau \xi, 0)\); we obtain a compact space \( \Omega^* \). Let \((\xi, u) \rightarrow A(\xi, u) \) be a continuous function \( \Omega^* \rightarrow \mathbb{C} \) such that \( A(\cdot, u) \in \mathcal{E}_u \) and \( u \rightarrow A(\cdot, u) \) is continuous from \([0,1]\) to \( \mathcal{E}_u \). We call \( \mathcal{E}_u^* \) the Banach space of such functions with norm

\[
\|A\|_{\mathcal{E}_u^*} = \max_0^1 \|A(\cdot, u)\|_{\mathcal{E}_u}.
\]

We denote by \( \mathcal{E}_u^* \) the dual of this space.

Let \( \Theta \) be a real continuous and strictly positive function on \( \Omega^* \). The suspended flow with speed function \( \Theta \) is the flow \( \mathcal{H}_s^\Theta \) defined on \( \Omega^* \) by

\[
\mathcal{H}_s^\Theta(\xi, u) = (\xi, u(t)), \quad \frac{du(t)}{dt} = -\frac{1}{\Theta(\xi, u)},
\]

with appropriate identifications when \( u(t) = 0 \) or \( 1 \). This flow is mixing if there is no \( A \in \mathcal{E}_u(\Omega^*) \) satisfying \( A + \mathcal{H}_s^\Theta = e^{i\alpha A} \) with \( \alpha > 0 \), and \( (\mathcal{H}_s^\Theta) \) is nonmixing if and only if it is isomorphic to a flow with constant speed
function. The correspondence between $\Omega$ and $\Omega^w$ extends to invariant (probability) measures, and to functions, as follows:

$$\Omega^w \supset \{(\xi, u)\} \quad \sigma^w \rightarrow \sigma^w \quad \mathcal{A}$$

(see formulas (2.2), (2.3), and (2.6) which follow).

If $\sigma$ is a $\tau$-invariant measure on $\Omega$, a $(\xi^w)$-invariant measure $\sigma^w$ on $\Omega^w$ is defined by

$$\sigma^w(d\xi du) = \sigma(d\xi)\Theta(\xi, u) du,$$

where $du$ denotes Lebesgue measure. If $\sigma$ is a $\tau$-invariant probability measure, then a $(\xi^w)$-invariant probability measure $\sigma^w$ is given by

$$\sigma^w = \sigma^w\left(\int \sigma(d\xi) r(\xi)\right)^{-1},$$

where we have written

$$r(\xi) = \int \Theta(\xi, u) du.$$

The map $\sigma \mapsto \sigma^w$ is a bijection of the $\tau$-invariant probability measures on $\Omega$ to the $(\xi^w)$-invariant probability measures on $\Omega^w$. The entropy $h_{\xi^w}(\sigma^w)$ with respect to $(\xi^w)$ is given by Abramov's formula:

$$h_{\xi^w}(\sigma^w) = h(\sigma)\left(\int \sigma(d\xi) r(\xi)\right)^{-1}.$$ 

The pressure of $\mathcal{A} \in \mathcal{F}(\Omega^w, R)$ is defined by

$$P^w(\mathcal{A}) = \max\left\{ h_{\xi^w}(\sigma^w) + \sigma^w(\mathcal{A}) : \sigma^w \text{ is a $(\xi^w)$-invariant probability measure} \right\}.$$

Write

$$\mathcal{A}_w(\xi) = \int \mathcal{A}(\xi, u)\Theta(\xi, u) du.$$ 

Then $\mathcal{A}_w \in \mathcal{F}(\Omega, R)$. (Note that $1_w = r$ by (2.4)) If $\mathcal{A} \in \mathcal{W}^w$, there is a unique measure $\rho^w$ realizing the maximum in (2.5). This is called the Gibbs state for $\mathcal{A}$. In fact $\rho^w$ corresponds by (2.2), (2.3) to the $\tau$-invariant probability measure $\rho$ on $\Omega$ which is the Gibbs state for $\mathcal{A}_w - P^w(\mathcal{A})r$. Furthermore $P(\mathcal{A}_w - P^w(\mathcal{A})r) = 0$ and this equation determines $P^w(\mathcal{A})$ (see Bowen and Ruelle [4]).

3 For a precise statement see [1].
Let us return to the original Axiom A flow \( (f^t) \) on the manifold \( M \). The connection between the flow \( (\xi_i^t) \) on \( \Omega^* \) and \( (f^t) \) restricted to a basic set \( \Lambda \) of \( M \) is by a map \( \varphi: \Omega^* \to \Lambda \) (see Bowen [2]). The map \( \varphi \) sends \( (\xi, 0) \) to a point \( x_i \) of the hypersurface \( \Sigma \), such that its orbit successively intersects all \( \Sigma \), in the order given by the components \( \xi_i \) of \( \xi \). The point \( (\xi, u) = \xi_i^u(\xi, 0) \) goes to \( f^u x_i \). Using \( \varphi \), one can send functions on \( \Lambda \) to functions on \( \Omega^* \) and measures on \( \Omega^* \) to measures on \( \Lambda \). In this manner, the study of correlation functions for the Axiom A flow \( (f^t) \) translates into the study of correlation functions for the suspended flow \( (\varphi^t) \). This approach, called symbolic dynamics, has the disadvantage of a certain arbitrariness (the choice of \( \Omega^* \), \( \varphi \) is nonunique) but we shall not further consider the question. (For Axiom A diffeomorphisms, the problem has been discussed in [9], and one could repeat the same remarks here, mutatis mutandis.)

Note that the positive function \( r \) on \( \Omega \) defined by (2.4) expresses the time between crossings \( S - \Lambda \) and the next hypersurface \( \Sigma \) in terms of the symbol sequence. By suitably choosing the hypersurfaces \( \Sigma \) (they should be anisotropes of unstable manifolds) one can assume that \( r(\xi) \) depends only on the components \( \xi_i \) of \( \xi \) with \( k \leq 1 \).

We thus have

\[
\rho \circ r^{-1} = \rho \circ \sigma_{\varphi^{-1}(0)},
\]

where \( \rho \) is a function on \( \Omega \) and \( \sigma_{\varphi^{-1}(0)}: \Omega \to \Omega \) has been defined in (1). From the general theory of Axiom A flows (see Bowen [2], Bowen and Ruelle [4]), it follows that the time between crossings of the hypersurfaces \( \Sigma \) is a Hölder continuous function and, as a consequence, that \( r \) belongs to \( \Psi_\theta^{-1} \) for suitable \( \theta \). Similarly, if \( A \) is a smooth function on the manifold \( M \), and we define \( A = A \circ \varphi \) and \( A_r \) by (2.6) we find \( A_r \in \Psi_\theta \) for suitable \( \theta \) (for technical reasons we want \( \theta^2 \) here rather than \( \theta \)).

From now on we shall work with the symbolic dynamics, remebering from the differentiable setup only that \( A, \varphi \in \Psi_\theta^2 \), so that

\[
\rho \in \Psi_\theta((-\infty, 0]), \quad A_r \in \Psi_\theta^2
\]

follow from (2.4), (2.6).

Remember that an interaction \( \Phi \) has been associated with \( A_r \) by (1.1). It is convenient to introduce also an interaction \( \Psi \), associated with the function \( \rho \) defined by (2.7), such that

\[
r(\xi) = -\Psi(\xi) - \sum_{i=1}^n \Psi \left( \xi_{i+1}, \ldots, \xi_n \right)
\]

and \( \psi \psi \leq \text{const} \theta^2 \). Note that with the notation of (1.1) we have \( r = A_r \).
For real $A \in \mathbb{R}^n$, a zeta function is defined by

$$
\zeta_A(s) = \prod_{r=1}^{n} \left( 1 - \exp \left[ s \left( \frac{1}{2} \right) \frac{A_r}{d} \right] \right) \frac{1}{(1 - s)}
$$

where the product is over the periodic orbits $\gamma$ for the flow, $x_r \in \gamma$, and $\zeta(s)$ is the prime period of $\gamma$ (Ruelle [5]); the functional $d$ is defined by (1.4). The expression (2.2.6) converges and $\zeta_A(s)$ is analytic for $\Re s > \rho(s, A)$. The poles of $\zeta_A(s)$ are located at the points $s$ such that $1$ is an eigenvalue of $d^{-1} A d^{-1}$.

2.1. Theorem. (a) (Pollicott [6]) $\zeta_A(s)$ extends to a meromorphic function in $s: \Re s > \rho(s, A) - \delta$, where $\delta$ is determined by $\rho(s, A) - \delta = \log \delta$. The poles of $\zeta_A(s)$ are located at the points $s$ such that $1$ is an eigenvalue of $d^{-1} A d^{-1}$.

(b) (Ruelle [8]) $\zeta_A(s)$ has a simple pole at $s = \rho(s, A)$.

(c) (Parry and Pollicott [5]) If $\zeta_A(s)$ is missing, $\zeta_A(s)$ has no pole on the line $s: \Re s = \rho(s, A)$ apart from the pole at $s = \rho(s, A)$.

By analogy with the proof of the prime number theorem one can, in view of (c), apply the Wiener-Ikehara Tauberian theorem to $\zeta_A(s)$ to study the distribution of the periods $\gamma(\gamma)$ (Parry and Pollicott [5]).

It may be convenient to consider a functional defined with respect to the flow $(\gamma)_{\tau}$ with unit speed. For $A \in \mathbb{R}^n$, define

$$
\mathcal{S}(A) = \prod_{r=1}^{n} \left( 1 - \exp \left[ s \left( \frac{1}{2} \right) \frac{A_r}{d} \right] \right) \frac{1}{(1 - s)} - d(A_{\tau}),
$$

where $A_{\tau} = \int_0^\tau A(\xi, u) du$, and $\zeta_A(s)$ is the (integer) period of $\gamma$ with respect to $\gamma$. With this definition, $\zeta_A(s) = \mathcal{S}(A - s \theta)^{-1}$. The function $A \rightarrow \mathcal{S}(A)$ is holomorphic on $\mathbb{R}^n$, when $P(Re A + \log \theta) < 0$; the function $A \rightarrow \mathcal{S}(A)$ is holomorphic on $\mathbb{R}^n$ when $P(Re A + \log \theta) < 0$.

3. Gibbs distributions for the flow $(\gamma)_{\tau}$

The concept of Gibbs distributions for a lattice system introduced in [5] was shown to be a natural extension of the concept of Gibbs state. If we want to study Axiom A flows rather than diffeomorphisms, we need another concept.

The definition presented here is somewhat ad hoc, but appropriate for the discussion of correlation functions as we shall see later. It is in fact a natural continuous time version of the concept introduced earlier for discrete systems, but restricted to the simplest case (see Remark below).
For discrete systems, a space $\mathcal{S}_n$ of Gibbs distributions on $\Omega$ is defined as the span of elements of the form

$$e^\langle d\psi \rangle^\sigma(e^{-\gamma E(V,V')})e^{-\gamma E(V,V')}.$$ 

In this formula, $\sigma$, $\gamma$ belong to the generalized eigenspaces to the eigenvalues $\lambda_j$, $\mu_j$ of operators $\mathcal{L}_\psi^{\mathcal{S}_n}$, $\mathcal{L}_{\mathcal{S}_n}$ acting on $\mathcal{H}_\psi^\sigma((-\infty,0])$ and $\mathcal{H}_\psi^\sigma([1,\infty])$, respectively. The operator $\mathcal{L}_\psi^{\mathcal{S}_n}$ is the dual of $\mathcal{L}_\psi^\mathcal{S}_n$ defined by (1.3), and $\mathcal{L}_{\mathcal{S}_n}$ is defined analogously. We have written

$$\mathcal{V}_\nu(\xi',\nu') = \sum_{\nu=0}^\infty \sum_{\nu=1}^\infty \Phi_{\nu,\nu}(\xi',\xi',\ldots,\xi',\xi',\ldots,\xi').$$

Thus, all the ingredients of $\mathcal{S}_n$ are defined with respect to an interaction $\Phi$, or equivalently a function $A_k$ (see (1.3)). (Note that $A_k = \tau^{4k}$ is, up to a sign, the contribution to the energy of the lattice site $k$ for the standard interpretation of the formalism we are describing.)

Our first step towards a definition of Gibbs distributions for continuous time systems will be to replace $A_k = \tau^{4k}$ by different functions for $k < 0$ and $k > 1$. Most precisely, we replace $A_k$ by $A_k - \tau^{r^k}$ for $k < 0$ and $A_k - \tau^{r^k}$ for $k > 1$. (We start from real $\theta$, $\theta \in \mathbb{R}$, with $\theta > 0$, and $r$, $A_k$ are defined by (2.4), (2.5). The complex numbers $v$, $w$ will be specified in a minute.) Using the interactions $\Phi$, $\Psi$ defined by (1.3), (2.8) and the definitions (1.2), (1.3) we see that the operator $\mathcal{L}_\mathcal{S}$ associated with $A_k - \tau^{r^k}$ is $\mathcal{L}_\mathcal{S}^{\mathcal{S}_n}$ such that

$$(\mathcal{L}_\mathcal{S}^{\mathcal{S}_n} \psi)(\xi') = \sum_{\nu=0}^\infty \sum_{\nu=1}^\infty \exp(A_k - \tau^{4k}) \xi' \psi(\tau^{4k} \xi').$$

There is an analogous definition for $\mathcal{L}_\mathcal{S}^{\mathcal{S}_n}$. In the function $V_\psi$ defined by (3.1) we replace $\Phi$ by $\psi - \tau^\theta$ (not $\theta - \tau^\theta$, the reason for this asymmetric choice is that the difference between $A_k - \tau^{4k}$ and $A_k - \tau^{r^k}$, vert. $(-v - w)\tau^{r^k}$, depends only on arguments $\xi_k$ with $k \in (-\infty, 0]$).

The numbers $v$, $w$ are specified by the condition that (3) be an eigenvalue of $\mathcal{L}_\mathcal{S}^{\mathcal{S}_n}$ and $\mathcal{L}_\mathcal{S}^{\mathcal{S}_n}$, and that

$$(\mathcal{L}_\mathcal{S}^{\mathcal{S}_n} \psi)(\xi') = \theta < Re v, Re w,$$

where $\theta$ is determined by Theorem 2.1(a). (We have also automatically Re v, Re w < $P_n(A)$)

1 Equivalently, we might use $A_k - \tau^{4k}$ for $k < 0$ and $A_k - \tau^{r^k}$ for $k > 1$; the final definitions would not change.
Let $F_u^*$ and $F_u^{**}$ be the eigenspaces to the eigenvalue 1 of $\mathcal{U}_{\text{to}}$ and $\mathcal{U}_{\text{toe}}$. (Note: the strict, not the normalized eigenspaces.) We let $\mathcal{F}_{\text{to}}$ be the finite dimensional subspace of $\mathcal{Y}_2$ generated by the elements

$$\sigma_u(dU^v \wedge dU^r) = \sigma_u(dU^v)\sigma_u(dU^r)\exp[-(v-w)\xi(r')],$$

where $\sigma_u' \in F_u^*$, $\sigma_u^{**} \in F_u^{**}$. (It is not hard to see that $F_u^* \oplus F_u^{**} \rightarrow \mathcal{F}_{\text{to}}$ is bijective.)

The restriction of $\mathcal{U}_{\text{to}}$ to $F_u^*$ is the identity operator; similarly for the restriction of $\mathcal{U}_{\text{toe}}$ to $F_u^{**}$. Using (3.3), it is now readily checked that

$$\{\tau_{\sigma_u'}(dU^v \wedge dU^r) = \sigma_u^{**}(dU^v)\sigma_u^{**}(dU^r)\exp[-(v-w)\xi(r')] - \mathcal{V}_{\text{to}e}(\xi \wedge \xi')\}$$

so that, for all $\sigma_u \in \mathcal{F}_{\text{to}}$, 

$$\tau_{\sigma_u} = \exp[(v-w)r - \tau] \cdot \sigma_u,$$

or equivalently

$$\tau^{-1}\sigma_u = \exp[(v-w)r] \cdot \sigma_u.$$

Define now $\sigma^* \in \mathcal{Y}_{2*}$ by

$$\sigma_u(dU^v) = \sigma_u(dU^v) \cdot \exp[-(v-w)r] \cdot \Theta\{\xi, u\},$$

where $r = r(\xi, u)$ is the inverse of the function $t \mapsto u(t)$ such that $du/dt = \Theta(\xi, u)$ and $u(0) = 0$, i.e., $r(\xi, u) = \int_0^{u} d\theta(\xi, \sigma)$ (Note that $r(\xi, 1) = r(\xi)$ and that $|\theta|$ $\leq$ $\theta$ in view of (2.1).) We define the space $\mathcal{Y}_{2*}$ of Gibbs distributions to consist of the $\sigma^*$ constructed above.

Writing $(\sigma^* a)(\xi) = a(\xi + \Theta\{\xi, u\})$ we find that

$$\frac{d}{dt} \sigma^* a = -(v-w)\sigma^* a$$

for $\sigma^* \in \mathcal{Y}_{2*}$, hence

$$\sigma^* a = \exp[(v-w)\xi].$$

Returning to the space $\mathcal{F}_{\text{to}}$, we note that the projection $\pi_{\xi \wedge \xi'}$ is readily characterized. We have indeed, from (3.3)

$$\pi_{\xi \wedge \xi'}(dU^v) = S_{\xi}(\xi') \sigma_u(dU^r),$$

where

$$S_{\xi}(\xi') = \int_\xi \sigma_u^{**}(dU^r) \exp[-\mathcal{V}_{\text{to}e}(\xi \wedge \xi')] .$$
It is known (see [9]) that the functions $S_{-\sigma}$ of this form (with $\pi_{-\sigma} \in F_{-\sigma}$) constitute precisely the eigenspace $F_{-\sigma}$ to the eigenvalue 1 of the operator $L_{-\sigma}^{*}$ acting on $\Psi_{\sigma}$ ($-\infty, 0]$. Thus $\pi_{-\sigma} \in F_{-\sigma}$ is spanned by $F^{*}_{-\sigma}$. Note that we also have
\[ \pi_{-\sigma} \otimes \sigma_{\sigma} = \exp \left[ (\varphi - \varphi') \right] \cdot \pi_{-\sigma} \otimes \sigma_{\sigma}. \]

**Example.** Since $P(A_{\sigma} - P^{*}(A_{\sigma})) = 0$ (52), the operator $\mathcal{L}_{0}^{*} \rho_{\sigma_{\sigma}}$ has 1 as simple eigenvalue (see §1). Writing $P^{*}(A) \in \Omega$, we see that $\mathcal{F}_{\rho_{\sigma}}^{*}$ is one-dimensional spanned by the Gibbs state $\sigma^{*}$ on $\Omega$ for $A_{\sigma} - P^{*}(A_{\sigma})$. The space $\mathcal{F}_{\rho_{\sigma}}^{*}$ is thus spanned by the Gibbs state $\rho^{*}$ on $\Omega_{0}$ for $A_{\sigma}$, and $\rho^{*}$ is therefore also a Gibbs distribution.

**Remark.** To avoid technical problems we have adopted a definition of Gibbs distributions which uses the strict eigenspaces of $L_{\sigma}^{*}$, $L_{-\sigma}^{*}$. (This is no restriction as long as we consider simple eigenvalues.) The parallel study in [9] was based on a more comprehensive definition, using generalized eigenspaces. As a consequence we could identify in [9] all the coefficients of the poles of the Fourier transform of correlation functions. Here we identify the residues in terms of Gibbs distributions for the important case of simple poles. A more general analysis would of course be desirable.

**Example.** Let $r$ be a constant function, say $r = T$. Then $\mathcal{L}_{-\sigma}^{*} = e^{-rT} \mathcal{L}_{-\sigma}$, $\mathcal{L}_{\sigma}^{*} = e^{-rT} \mathcal{L}_{\sigma}^{*}$. The eigenvalues and eigenvectors are thus readily determined. In particular, 1 is an eigenvalue of $\mathcal{L}_{0}^{*}$ if $\lambda \in e^{-rT} = 1$, where $\lambda$ is an eigenvalue of $\mathcal{L}_{0}^{*}$.

This gives
\[ \varphi = \varphi \frac{1}{T} (\log \lambda + 2k \pi i), \]
where the multivaluedness of the log has been made explicit. Writing similarly
\[ \varphi = \varphi \frac{1}{T} (\log \lambda + 2k \pi i) \]
we may identify $\sigma^{*} \in \mathcal{F}_{\sigma}$ with an element of the space $\mathcal{F}_{\sigma}$ of Gibbs distributions defined in [9]. The corresponding $\sigma^{*} \in \mathcal{F}_{\sigma}^{*}$ is given by
\[ \sigma^{*}(d\xi) = \exp \left[ (\varphi - \varphi') \left( \frac{1}{T} \log \lambda + 2k (k' - k) \pi i \right) \right], \]
and we have
\[ \pi_{\sigma} \sigma^{*} = \sigma^{*} \exp \left[ (\varphi - \varphi') \left( \frac{1}{T} \log \lambda + 2k (k' - k) \pi i \right) \right]. \]
Notice that $\mathcal{F}_{\sigma}^{*} = \mathcal{F}_{\sigma}^{*}$ when $\nu' - \nu = \nu' - \nu = l \cdot 2 \pi i / T$, $l$ an integer.
4. Correlation functions for the flow \( \{ \mathcal{Q}_k \} \)

We consider the suspended flow for a fixed speed function \( \Theta^{-1} \in \mathfrak{g}_o^* \), and let \( \rho \) be the Gibbs state corresponding to the real function \( A \in \mathfrak{g}_o^* \). If \( B, C \in \mathfrak{g}_o^* \) we define

\[
\rho_{BC}(t) = \rho^*(B \cdot \mathcal{Q}_t) \cdot C - \rho^*(B) \rho^*(C)
\]

and its Fourier transform

\[
\hat{\rho}_{BC}(\omega) = \int_{\mathbb{R}^+} e^{i\omega t} d\rho_{BC}(t)
\]

which has to be understood as a tempered distribution. We may express \( \rho^* \) in terms of the Gibbs state \( \rho \) for \( A_o \) on \( \mathfrak{g} \) (see (2.2), (2.3)). We write

\[
\rho^* = \int \rho(\mathcal{Q}) \int \Theta(\xi, u) du - \int r(\xi) \rho(\mathcal{Q})
\]

and

\[
B' = B - \rho^*(B), \quad C' = C - \rho^*(C)
\]

The following manipulations then yield a correct result in the sense of distributions

\[
\hat{\rho}_{BC}(\omega) = \int_{\mathbb{R}^+} e^{i\omega t} d\rho^* \left((B' \cdot \mathcal{Q}_t)C'\right)
\]

\[
- \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \int \Theta(\xi, v) du B'(\mathcal{Q}_t(\xi, u)) C' (\xi, u)
\]

\[
= \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \int C' (\mathcal{Q}_t(\xi, 0)) \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right]
\]

\[
= - \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right] - \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right]
\]

\[
= - \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \sum_{k=0}^{\infty} e^{i\omega k} \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right]
\]

\[
= - \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \sum_{k=0}^{\infty} e^{i\omega k} \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right]
\]

\[
= - \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \sum_{k=0}^{\infty} e^{i\omega k} \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right]
\]

\[
= - \int_{\mathbb{R}^+} e^{i\omega t} d\rho(\mathcal{Q}_t) \sum_{k=0}^{\infty} e^{i\omega k} \left[ \Theta(\xi, 0) \cdot du B'(\mathcal{Q}_t(\xi, 0)) \right]
\]
where
\[ h(\xi, u) = \int_0^1 e^{i\omega t} dB (\frac{\xi}{\omega}(\xi, 0)) \]
\[ = \int_0^1 d\theta B(\xi, u) B^\prime(\xi, u) e^{i\omega t}(\xi, u). \]
\[ \tau(\xi, u) = \int_0^\infty d\theta d\phi(\xi, n). \]

The definition of \( \hat{C} \) is similar.

Note that, for each \( \sigma > 0, \sup_{\omega \in \Omega^2 \setminus \Omega^2_{\omega_0}}, \sup_{\xi, \omega} | \hat{C}(\xi, \omega) | \) are bounded uniformly with respect to \( \omega \) and \( \xi \). Using furthermore the fact that \( \min \Gamma = 0 \), we see that the right-hand side of (4.2) converges in Schwartz’ space \( S'(\mathbb{R}) \) of temperate distributions (with respect to \( u \)) and thus also in the space \( S'(\mathbb{R}) \) of all distributions. We shall next represent \( \hat{B}, \hat{C} \) as series which converge uniformly on compacts, as well as their derivatives:

\[ \hat{B} = X_0 + X_1 e^{i\omega r} + X_2 e^{i2\omega r} + \cdots + X_{m-1} e^{im\omega r} + \cdots + \]
\[ \hat{C} = Y_0 + Y_1 e^{i\omega r} + Y_2 e^{i2\omega r} + \cdots + Y_{m-1} e^{im\omega r} + \cdots. \]

We define successively \( B_0 = B, X_0, B_1, Y_1, \ldots \) as follows:

(a) Treating \( \omega \) as a parameter, which we now allow to be complex, we extract \( X_0 \) as the part of \( B_0 \) depending only on \( \xi_{m-1}, \xi_m \). This extraction is not unique, but can be achieved linearly, so that

\[ \text{var}_{m+1} X_m = 0, \quad \| X_m \|_{\omega} \leq \| B_m \|_{\omega} \quad \left| B_m - X_m \right|_{\omega} \leq \text{var}_{m+1} B_m. \]

(b) We define

\[ B_{m+1} = (B_m - X_m) e^{-i\omega r}. \]

We may assume that

\[ \text{var}_{m+1} B_m \leq K \delta^k, \quad \| B_m \|_{\omega} \leq K, \quad \text{var}_{m+1} R_m \leq L \delta^k \quad \text{for} \quad k \geq 1. \]

(\( K \) depends on \( u \), but is uniformly bounded on compacts; from (4.3) and (4.4) we see that we may take \( K = \| (\Theta)^2 \|_{\omega} \| B \|_{\omega} \| \exp(\omega u) \|_{\omega} \| \Theta \|_{\omega} \). We also have \( L = |r\|_{\omega} \| (\Theta)^2 \|_{\omega} \).) Note that by construction we have

\[ \text{var}_{m+1} (B_m - X_m) \leq \text{var}_{m+1} B_m \quad \text{for} \quad k \geq m. \]

In view of (4.6), (4.7), (4.8), (4.9), we obtain

\[ \text{var}_{m+1} B_m \leq K \delta^k \quad \text{for} \quad k \geq m \]

provided the \( K_m \) satisfy \( K_0 \geq K \) and

\[ K_{m+1} \geq \frac{1}{2} (K_m + \text{var}_{m+1} B_m \delta^{k+1}) \]
with
\[ E = \begin{cases} \exp(i\omega \cdot \max r) & \text{for } |\omega| > 0, \\ \exp(i\omega \cdot \min r) & \text{for } |\omega| < 0. \end{cases} \]

We take
\[ K_m = \bar{K} E^m, \quad \bar{K} = \mathcal{K} \prod_{k=1}^m (1 + |\omega|^m \theta^{2k+1}). \]

Thus
\[ \text{Var}_k B_m \leq \bar{K} E^m \theta^k \text{ for } k \geq m, \]
\[ ||B_m - X_m||_\infty \leq \bar{K} E^m \theta^{m+1}. \]

Similar estimates hold for the derivatives of the \( X_m \) with respect to \( \omega \). Therefore for \( \omega \) real, and thus \( E = 1 \), the series (45) for \( \bar{K} \) and the differentiated series converge exponentially fast on compact sets. In the sense of convergence in \( \mathcal{B}(R) \) we therefore have
\[ \rho_{2m}^{\bar{K}}(\omega) \]
\[ = \int \rho(d\xi) \sum_{j=-m}^m \sum_{k=0}^{j-1} r(\tau^j \xi) \]
\[ \cdot \sum_{n=0}^\infty X_n(\tau^n \xi, \omega) \exp(\omega r(\tau^n \xi) + \cdots + \omega r(\tau^{n-j-1} \xi)) \]
\[ \cdot \sum_{s=0}^\infty Y_s(\tau^s \xi - \omega) \exp(-i\omega r(\xi) + \cdots + \omega r(\tau^{-s} \xi)) \]
\[ - \int \rho(d\xi) \sum_{m=0}^\infty \sum_{j=-m}^m X_m(\tau^{-j} \xi, \omega) Y_j(\tau^{-j} \xi, -\omega) \exp(\omega r(\tau^{-j} \xi) + \cdots + \omega r(\tau^{-j-m-1} \xi)) \]
\[ + \sum_{n=0}^\infty \left( \exp(\omega r(\tau^n \xi)) \theta^n(\tau^n \xi, \omega) C^n(\tau^n \xi, -\omega) \right) \]
where
\[ B^\omega(\xi, \omega) = \sum_{a=0}^{\infty} X_a(\tau^a \xi, \omega), \]
\[ C^\omega(\xi, -\omega) = \sum_{a=0}^{\infty} Y_a(\tau^a \xi, -\omega). \]

We may write \( \tilde{\rho} = \tau_{\omega, 0} \rho \tau_{\omega, 0}^{-1} \), and
\[ \tau_{\omega, 0} = \tau_{\omega, 0} \tau_{\omega, 0}^{-1} = \tau_{\omega, 0} \tau_{\omega, 0}^{-1}, \]
\[ B^\omega(\xi, \omega) = \tilde{B}_\omega \in C_{\omega, 0} \xi \]
\[ C^\omega(\xi, -\omega) = \tilde{C}_\omega \in C_{\omega, 0} \xi \]
so that
\[ \tilde{\rho}_{\tilde{\omega}}(\omega) = \rho(\omega) \left( \sum_{i=0}^{\infty} \exp \left( i \omega \sum_{k=0}^{i-1} (\tau^k \xi) \right) \tilde{B}_\omega(\tau^i \xi) \tilde{C}_\omega(\tau^i \xi) \right. \]
\[ + \sum_{i=0}^{\infty} \exp \left( i \omega \sum_{k=0}^{i-1} (\tau^k \xi) \right) \tilde{B}_\omega(\tau^i \xi) \tilde{C}_\omega(\tau^i \xi) \]
\[ \left. - \tilde{B}_\omega(\tau^i \xi) \tilde{C}_\omega(\tau^i \xi) \right). \]

We have (see (3.6))
\[ \rho(\omega) = \tilde{S}_{\omega}(\omega) S_{\omega}(\omega) d\omega, \]
where \( P = P_{1\omega}(4) \) and \( S_{\omega} \) and \( \tilde{S}_{\omega} \) are the eigenvectors corresponding to the eigenvalue 1 of the operators \( X_{\omega,p} \) acting on \( \mathcal{H}_p(\omega, 0) \). (These eigenvectors are unique up to normalization.) Thus
\[ \tilde{\rho}_{\tilde{\omega}}(\omega) = \rho S_{\omega}(\omega) \left( \tilde{B}_\omega \sum_{i=0}^{\infty} \exp \left( -i \omega \sum_{k=0}^{i-1} \left( S_{\omega} B_{\omega} \right) \right) \tilde{S}_{\omega}(\omega) B_{\omega} \right) \]
\[ + \tilde{C}_\omega \sum_{i=0}^{\infty} \exp \left( -i \omega \sum_{k=0}^{i-1} \left( S_{\omega} C_{\omega} \right) \right) \tilde{S}_{\omega}(\omega) C_{\omega} \]
\[ - \tilde{B}_\omega \tilde{S}_{\omega}(\omega) \tilde{C}_{\omega} \tilde{S}_{\omega}(\omega) \]
\[ = \rho \left[ \tilde{B}_\omega \left( 1 - S_{\omega} \tilde{S}_{\omega} \right) S_{\omega} \tilde{S}_{\omega} \left( 1 - S_{\omega} \tilde{S}_{\omega} \right) \tilde{B}_\omega \right] - \rho \left( \tilde{B}_\omega \tilde{C}_\omega \right) \]
\[ = \rho \left[ \tilde{B}_\omega \left( 1 - S_{\omega} \tilde{S}_{\omega} \right) S_{\omega} \tilde{S}_{\omega} \left( 1 - S_{\omega} \tilde{S}_{\omega} \right) \tilde{B}_\omega \right] \frac{1}{2} \tilde{B}_\omega \]
\[ + \rho \left[ \tilde{C}_\omega \left( 1 - S_{\omega} \tilde{S}_{\omega} \right) S_{\omega} \tilde{S}_{\omega} \left( 1 - S_{\omega} \tilde{S}_{\omega} \right) \tilde{C}_\omega \right] \frac{1}{2} \tilde{C}_\omega \]
Note that the two terms in the right-hand side are permutea by the interchange of \( B \) and \( C \), and the replacement of \( \omega \) by \(-\omega\).

**4.1. Theorem.** If \( B, C \in \mathbb{F}^{\omega} \), the function \( \hat{\rho}_C^B \) extends to a meromorphic function in the strip

\[ \text{Im} \omega < \theta^* \],

where

\[ \theta^* = \frac{\log \theta}{2 \max r - \min r} \].

If we also have \( \text{Im} \omega < \delta \), we may write

\[ \hat{\rho}_C^B (\omega) = \frac{N_C (\omega)}{d \left( \text{exp} \left( A_\omega - (P^\omega(A) + i\omega) r \right) \right)} + \frac{N_C (-\omega)}{d \left( \text{exp} \left( A_\omega - (P^\omega(A) - i\omega) r \right) \right)} \]

(14.14)

where \( N_{AC} \) is holomorphic in (14.13) and \( d \) is as in (14.4).

Note that, in view of (2.9), we may rewrite (14.4) as

\[ \hat{\rho}_C^B (\omega) = N_C (\omega) \left( P^\omega(A) + i\omega \right) + N_C (-\omega) \left( P^\omega(A) - i\omega \right) \].

The position of the poles of \( \hat{\rho}_C^B \) is thus simply related to that of the poles of \( \hat{s} \).

(Are they of the form \( \pm (P^\omega(A) - s) \), where \( s \) is such that 1 is an eigenvalue of \( \hat{\Delta}_n \)?

A partial proof of the above proposition has been obtained earlier by Pollicott [7].

Let \( \theta < \theta^* < 1 \); then \( \omega \rightarrow \hat{\rho}_C^B \) is holomorphic with value in \( \mathbb{F}(-\infty, 0) \) in the region defined by \( \theta^{*+1} \text{Re} \theta = 1 \), i.e.,

\[ 2|\log \theta^*| < \left| \log \theta \right| - \text{Im} \omega \cdot \max r \quad \text{if} \quad \text{Im} \omega \geq 0, \]

\[ 2|\log \theta^*| < \left| \log \theta \right| - \text{Im} \omega \cdot \min r \quad \text{if} \quad \text{Im} \omega < 0, \]

On the other hand, \( 1 - \frac{\gamma}{\Delta_\infty (\omega \rightarrow \omega)} - \frac{\partial^\omega \gamma}{\Delta_\infty (\omega \rightarrow \omega)} \) is meromorphic as an operator on \( \mathbb{F}(-\infty, 0) \) provided

\[ \theta^* \exp \left( \text{Re} \left( A_\omega - (P^\omega(A) - i\omega) r \right) \right) < 1 \]

i.e.,

\[ P \left( A_\omega - (P^\omega(A) + i\omega) r \right) < \left| \log \theta^* \right| \].
Since \( P(A_r - P^* (A) r) = 0 \), this condition is implied by
\[
-\Im \omega \cdot \max \ r < \log \theta \phi.
\]
Therefore \( \omega \to (1 - S^{-1}(\sigma_{\nu}^+ r - (\sigma_{\nu}^+ r)^{-1}) \sigma_{\nu}^+ r)^{-1} \) is meromorphic if
\[
\frac{\log \theta}{\max \ r} < \frac{\log \theta}{\min \ r} < \frac{\log \theta}{\max \ r}
\]
and \( \omega \to \theta \sigma_{\nu}^{-1}(1 - S^{-1}(\sigma_{\nu}^+ r - (\sigma_{\nu}^+ r)^{-1}) \sigma_{\nu}^+ r)^{-1} \) is also meromorphic. Interchanging \( B \) and \( C \), we obtain from (4.12) the meromorphy of \( \sigma_{\nu}^{-1} \) in (4.13).

If \( \Im \omega < 0 \), we may also write
\[
(1 - S^{-1}(\sigma_{\nu}^+ r - (\sigma_{\nu}^+ r)^{-1}) \sigma_{\nu}^+ r)^{-1} = d \exp (A - (P(A) - i \omega) r)
\]
where the numerator is holomorphic in (4.15) (see [9, Proposition 3.2]); from this (4.14) follows readily.

4.2. Theorem. Suppose that \( 1 \) is a simple eigenvalue of \( \sigma_{\nu}^+ \). There is thus a simple eigenvalue \( \lambda(z) \) of \( \sigma_{\nu}^+ \) depending analytically on \( z \) for \( z \) close to \( 1 \). Assume that the derivative \( \lambda'(s) \neq 0 \). Then \( \sigma_{\nu}^{-1} \) has simple poles at \( \pm i(P(A) - s) \). Their residues are
\[
\frac{i}{K} \sigma_{\nu}^+ (B) \sigma_{\nu}^+ (C) \quad \text{and} \quad -\frac{i}{K} \sigma_{\nu}^+ (C) \sigma_{\nu}^+ (B)
\]
respectively, with \( \sigma_{\nu}^+ \in \mathbb{C}^n, \sigma_{\nu}^+ \in \mathbb{C}^n \), and \( K \) a constant.

(The normalization of \( \sigma_{\nu}^+ \), \( \sigma_{\nu}^+ \), and the value of \( K \) are discussed in the Remark of this paper.)

The two poles come from the two terms in the right-hand side of (4.12). It suffices to discuss the first term, which we rewrite
\[
\sigma_{\nu}(B) \left[ (1 - \sigma_{\nu}^+ (r - (\sigma_{\nu}^+ r)^{-1}) \sigma_{\nu}^+ r)^{-1} - \frac{1}{\lambda} \right] \sigma_{\nu}(C).
\]
Up to a contribution regular at \( \lambda(P(A) - s) \) this is
\[
\sigma_{\nu}(B) \left( \lambda^{-1} (1 - \lambda(P(A) - s)) \sigma_{\nu}(C) \right) \sigma_{\nu}(C)
\]
or, again up to a regular contribution,
\[
\sigma_{\nu}(B) \left( \lambda^{-1} (1 - \lambda(P(A) - s)) \sigma_{\nu}(C) \right) \sigma_{\nu}(C) \sigma_{\nu}(C) \left( S^{-1}(\sigma_{\nu}^+ r - (\sigma_{\nu}^+ r)^{-1}) \sigma_{\nu}^+ r)^{-1} \right) \sigma_{\nu}(C)
\]
\[
\sigma_{\nu}(B) \left( \lambda^{-1} (1 - \lambda(P(A) - s)) \sigma_{\nu}(C) \right) \sigma_{\nu}(C) \left( S^{-1}(\sigma_{\nu}^+ r - (\sigma_{\nu}^+ r)^{-1}) \sigma_{\nu}^+ r)^{-1} \right) \sigma_{\nu}(C)
\]

(4.16)
where we have used (3.6), (4.11), and \( \sigma_0 \in \mathcal{F}_p \), \( \sigma_0 \in \mathcal{F}_p \). We have, using successively (4.10), (4.4), (4.5), (4.3), and (5.5),

\[
\sigma_p \left( B^*(\cdot, i(P^*(A) - s)) \right) \\
= \sigma_p \left( \sum_{m=0}^{\infty} X_m \left( r^{-m}, i(P^*(A) - s) \right) \right) \\
= \sum_{m=0}^{\infty} \left( \tau^{-m} \sigma_p \right) \left[ X_m \left( r^{-m}, i(P^*(A) - s) \right) \right] \\
= \sigma_p \left( \sum_{m=0}^{\infty} X_m \left( r^{-m}, i(P^*(A) - s) \right) \exp \left( -P^*(A) - s \right) \right) \\
= \sigma_p \left( B \left( \cdot, i(P^*(A) - s) \right) \right) = \sigma_p^\# \left( B \right) = \sigma_p^\# \left( \mathcal{F}^* \right)
\]

with \( \sigma_p^\# \in \mathcal{F}_p^* \). Similarly

\[
\sigma_p \left( C^*(\cdot, -i(P^*(A) - s)) \right) = \sigma_p^\# \left( C \right)
\]

with \( \sigma_p^\# \in \mathcal{F}_p^* \). Inserting (4.17) and (4.18) in (4.16), we obtain

\[
i \lambda \left( x(t) \right) \sigma_p^\# \left( B \right) \sigma_p^\# \left( C \right) \left( \omega - i(P^*(A) - s) \right)^{-1}
\]

which is the form of the residue announced in the theorem, with \( K = \lambda^\#(s) \).

**Remark.** The product \( \sigma_p^\# \left( B \right) \sigma_p^\# \left( C \right) \) is unambiguously normalized in view of the formulas

\[
\sigma_p^\# \left( d\xi \right) = \sigma_p \left( d\xi \right) \exp \left( -P \right) dt,
\]

\[
\sigma_p^\# \left( d\xi \right) = \sigma_p \left( d\xi \right) \exp \left( -P \right) dt,
\]

\[
\left( \pi_{-n} \sigma_p \right) \left( d\xi^* \right) = S^{\#}_{\xi^*} \left( \xi^* \right) \sigma_p \left( d\xi^* \right),
\]

\[
\left( \pi_{-n} \sigma_p \right) \left( d\xi^* \right) = S^{\#}_{\xi^*} \left( \xi^* \right) \sigma_p \left( d\xi^* \right),
\]

\[
\sigma_p \left( S^{\#}_{\xi^*} \right) = 1, \quad \sigma_p \left( S^{\#}_{\xi^*} \right) = 1.
\]

The constant \( K \) is given by

\[
K = \lambda^\#(s) \left[ \int \sigma_p \left( d\xi^* \right) \right] \left[ \int \sigma_p \left( d\xi^* \right) \right],
\]

where \( \sigma_p \) is the Gibbs state \( \sigma_0 \in \mathcal{F}_p \) and \( \sigma_0 \in \mathcal{F}_p \), with

\[
\left( \pi_{-n} \sigma_p \right) \left( d\xi^* \right) = S^{\#}_{\xi^*} \left( \xi^* \right) \sigma_p \left( d\xi^* \right),
\]

\[
\left( \pi_{-n} \sigma_p \right) \left( d\xi^* \right) = S^{\#}_{\xi^*} \left( \xi^* \right) \sigma_p \left( d\xi^* \right),
\]

We obtained (4.19) from (4.3), and formula (3.2) of [9]. Note that \( K = 0 \) by one of the assumptions of Theorem 4.2.
References


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